A SPECIAL LIPSCHITZ MAP $f:\mathcal{C}\to H$ FOR THE HADAMARD SPACE H. SKETCH OF THE CONSTRUCTION

P. N. IVANSHIN

The work on this subject was motivated by [2]. So let H be an n-dimensional space of nonpositive curvature [1]. The problem is to find a Lipschitz function $f:\mathcal{C}\to H$ such that $f(\{x\})=\{x\}$ for any point $x\in H$. Assume that H is 1-connected and each two points of it can be connected with exactly one geodesic segment. The construction is inductive one. The construction for the 1-dimensional space H is evident.

1) Consider a point $p\in\partial_\infty H$ [3]. Consider a limit $d_p(x,y)=\lim\limits_{t\to\infty}d(x(t),y(t))$ for any pair of points x,y on a horosphere O_p^t. Each set $C\subset\mathcal{C}$ has a unique projection on O_p. Consider a problem for the limit space $\lim\limits_{t\to\infty}O_p(t),d_p$. Then take a first horosphere O_1 intersecting C and take an image of the limit point on it.

The only natural obstacle is the existence of the subsets on projections that shrink to a point in the limit. Note that any such subset is a closed subset of O_1.

2a) If the set C' shrinks as $t\to\infty$ then its preimage with respect to the projection onto the O_1 is convex set and has only one intersection point $p(C')$ with the appropriate horosphere $O_{C'}$.

2b) If it does not then the map $d_p(x,y)$ is a correctly defined metric. and there could be applied an induction step.

3) For the set of points $p_1,\dots p_n,\dots\subset O_1$ we apply a center mass construction from [**?**]. The only thing left to show is that this center depends on points with mass in Lipshitz way. To do this one must consider the convex set C of the fixed diameter D and mass $M(C)$. The problem now can be reduced to the case of the 0 curvature and the case of strictly negative one.

Lemma 1. *Let $p\in\partial_\infty H$ If there exists a geodesic segment on a horosphere O_p then the distance between any two points on this segment does not vanish for $t\to p$.*

• In case an horosphere possess a geodesic segment $T(x,y)$ there exists a Lambert rectangle of the type $\geq\pi/2$ then by assumption it must be $\pi/2$ or equivalently the geodesic lines do not converge one to another infinitely close, namely the distance between them equals length $d(x,y)$. $\triangleright$

Center of mass construction.

Lemma 2. *Consider a set $X=\bigcup\limits_{i=1}^{n}\{x_i\}\subset H$. The diameter of the convex hull* $\mathrm{diam}(Co(x_1,\dots,x_n))=\max\{d(x_i,x_j)|x_i,x_j\in X\}$.

Corollary 1. *Suppose for two sets $X=\bigcup\limits_{i=1}^{n}\{x_i\}$ and $Y=\bigcup\limits_{i=1}^{n}\{y_i\}$, $x_i,y_i\in H$ $\exists a>0$ the following: $\forall i,j\in\{1,\dots,n\}$ $\exists k,l\in\{1,\dots,n\}$ $d(y_i,y_j)\leq ad(x_k,x_l)$ then $a\mathrm{diam}(Co(x_1,\dots,x_n))\geq\mathrm{diam}(Co(y_1,\dots,y_n))$.*

Now define a center of the mass of the set of points with mass as follows: for two points $x_1, m_1,\ x_2, m_2$ this center is the point on the geodesic segment connecting x_1 to x_2 dividing the length of a segment in the proportion m_1/m_2. For n points one must consider first the center of the mass of the $n-1$ points $(x_2, m_2), \dots, (x_n, m_n)$ and then the first step of the construction for these points $(x_1^{'}, \frac{\sum_{i=2}^{n} m_i}{n-1}) \dots, (x_n^{'}, \frac{\sum_{i=1}^{n-1} m_i}{n-1})$. Consider then the same construction for a set of centers X_1 with the given masses. Then repeat the procedure.

Statement 1. *The center of the set of points with masses $X \subset H$ is correctly defined, i.e. it exists and is unique.*

Thus we can construct relatively good function for centers of mass which occur in nonsingular points. In the singular points this function will have discontinuities. To avoid this we must deform our function as follows: for all $x \in O_\infty$ is a center point for a set of the zero curvature such that the center of the mass of the whole set of points lies in the singular point $y \in O_\infty$ we must consider a function locally constant in the neighbourhood of this point, so that the center of mass stays in the point y. We need this to define a point on the fixed horosphere O_1 so that if the center depending on x equals y then the target point on O_1 lies in the subsequent shrinking set in the neighbourhood of the center point for this set.

The problem now reduces to the proof of the Lipshitz structure of a center mass function. To do this we must consider two different cases: 1) shift of one of the points of the set X; 2) change of the mass of one of the points of X. The proof is again by induction. It is trivial in case $|X| = 1$. Then for n points one has on the first step of the construction a set of points. Note that these points depend on the shift of one of them in Lipschitz way. Since this holds true on each subsequent construction step we can consider a converging series each element of which can be majored by an element from the standard Euclidean space $\mathbb{R}^n$ since the distance function for two points on distinct geodesic lines is convex [1].

References

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N.G. Chebotarev RIMM, KSU, Kazan, Russia
E-mail address: Pyotr.Ivanshin@ksu.ru